Effect of Higher Order Terms in Certain Nonlinear Finite Element Models

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Introduction

IT is not uncommon in finite element analyses of the nonlinear behavior of bars, plates, and shells to use lower order approximations for the additional terms in the potential energy expression which characterize the nonlinear behavior. Use of such mixed modes of approximation is often done on an ad hoc basis with more of an aim toward simplifying the numerical algorithm than for accuracy or consistency of the formulation. In fact, before a general development of criteria for consistent geometric stiffnesses was published,[1] many of the early studies of structural stability involved inconsistent geometric stiffness calculations which, nevertheless, gave satisfactory estimates of buckling loads.[2]

In the present Note, we investigate the effect of such a mixed formulation for a model problem of large transverse deflection of an elastic beam. We are able to show precisely when such mixed approximations can and cannot work for a representative class of problems. Moreover, we develop specific error estimates that enable us to state precisely the degree of accuracy lost (or gained) in such models, and the price one pays in accuracy for a gain in computational convenience. Indeed, we show that often the price is rather small.

Statement of the Model Problem

Consider a thin, homogeneous, simply supported, linearly elastic beam of cross-sectional area A and flexural rigidity EI under the action of compressive forces $P < P_{cr}$ and transverse forces $q(x)$. If $u(x)$ and $v(x)$ are the longitudinal and transverse components of displacement of particle on the beam's axis, if the effects of finite rotations are retained in the strain displacement relations, and if the Kirchhoff-Love (i.e., the Bernoulli) hypothesis is invoked, it is well-known that the total potential energy of the structure is

$$\pi(u, v) = \frac{1}{2}\int_0^L \left\{EIv''^2 + EAu'^2 + EAu'v'^2 + \frac{EA}{4}v'^4 - 2qv\right\} dx + Pu\big|_0^L \qquad (1)$$

Now we wish to use Eq. (1) to construct a finite element model of nonlinear behavior of the beam. We consider a beam segment of length h, between nodes 1 and 2 (say), and we introduce local approximations of the form

$$u(x) = \boldsymbol{\psi}(x)\mathbf{u}; \quad v(x) = \boldsymbol{\phi}(x)\mathbf{v} \qquad (2a, b)$$

wherein

$$\mathbf{u}^T = (u_1, u_2); \quad \mathbf{v}^T = [v_1, (dv/dx)_1, v_2, (dv/dx)_2]$$

$$\boldsymbol{\psi}^T(x) = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}; \quad \boldsymbol{\phi}^T(x) = \begin{Bmatrix} 1-3\xi^2+2\xi^3 \\ (\xi-2\xi^2+\xi^3)h \\ 3\xi^2-2\xi^3 \\ (-\xi^2+\xi^3)h \end{Bmatrix}; \quad \xi = x/h \qquad (3)$$

Clearly, the $\psi_i(x)$ are Lagrange interpolation polynomials and $\phi_j(x)$ are Hermite interpolation polynomials (our results are not restricted to these choices of functions).

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Now, instead of introducing Eq. (2) directly into Eq. (1) to obtain the energy for the element, one might use the linear interpolation functions $\boldsymbol{\psi}(x)$ to approximate terms in Eq. (1) involving $v'(x)$; e.g., for $v'(x)$ assume, instead of Eq. (2b)

$$v(x) \approx \eta(x) = \boldsymbol{\psi}(x)\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = [\psi_1, 0, \psi_2, 0]\mathbf{v} \equiv \bar{\boldsymbol{\psi}}(x)\mathbf{v} \qquad (4)$$

This leads to the approximate energy functional for the element

$$\pi_e(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\int_0^h \Big\{EI\mathbf{v}^T\boldsymbol{\phi}''^T\boldsymbol{\phi}''\mathbf{v} + EA\mathbf{u}^T\boldsymbol{\psi}'^T\boldsymbol{\psi}'\mathbf{u} + EA\mathbf{u}^T\boldsymbol{\psi}'^T\mathbf{v}^T\bar{\boldsymbol{\psi}}'^T\bar{\boldsymbol{\psi}}'\mathbf{v} + \frac{EA}{4}(\mathbf{v}^T\bar{\boldsymbol{\psi}}'^T\bar{\boldsymbol{\psi}}'\mathbf{v})^2 - 2q\mathbf{v}^T\boldsymbol{\phi}^T\Big\} dx + P\mathbf{u}^T\boldsymbol{\psi}^T\Big|_0^h \qquad (5)$$

and minimization of Eq. (5) leads to the following system of local nonlinear equilibrium equations

$$\tilde{\mathbf{K}}_v\mathbf{v} + \mathbf{Q}_1(\mathbf{u}, \mathbf{v}) + \mathbf{Q}_2(\mathbf{v}^3) = \mathbf{f}_v \qquad (6a)$$

$$\tilde{\mathbf{K}}_u\mathbf{u} + \mathbf{Q}_3(\mathbf{v}^2) = \mathbf{f}_u \qquad (6b)$$

where

$$\tilde{\mathbf{K}}_v = \int_0^h EI\boldsymbol{\phi}''^T\boldsymbol{\phi}''\,dx; \qquad \tilde{\mathbf{K}}_u = \int_0^h EA\boldsymbol{\psi}'^T\boldsymbol{\psi}'\,dx$$

$$\mathbf{Q}_1 = \int_0^h EA\bar{\boldsymbol{\psi}}'^T\bar{\boldsymbol{\psi}}'\mathbf{v}\boldsymbol{\psi}'\mathbf{u}\,dx; \quad \mathbf{Q}_2 = \int_0^h \frac{EA}{2}\bar{\boldsymbol{\psi}}'^T\bar{\boldsymbol{\psi}}'\mathbf{v}\mathbf{v}^T\bar{\boldsymbol{\psi}}'^T\bar{\boldsymbol{\psi}}'\mathbf{v}\,dx \qquad (6c)$$

$$\mathbf{Q}_3 = \int_0^h \frac{EA}{2}\boldsymbol{\psi}'^T\mathbf{v}^T\bar{\boldsymbol{\psi}}'^T\bar{\boldsymbol{\psi}}'\mathbf{v}\,dx$$

$$\mathbf{f}_v = \int_0^h q\boldsymbol{\phi}^T\,dx; \qquad \mathbf{f}_u = -P\boldsymbol{\psi}^T\Big|_0^h$$

The question is, of course, whether or not such an approximation will work and, if so, what effect will it have on the quality of the approximation? We shall answer this question by developing estimates of the error induced in the energy functional π by such approximations, assuming its second variation remains positive definite for the loading cases considered. We will also find it necessary to use certain interpolation results. Suppose $f(x)$ has derivatives of order $k+1$ which are square integrable on the interval $[0, L]$ and let $\tilde{F}(x)$ be a finite element interpolation function of $f(x)$ which coincides with $f(x)$ and derivatives of order $m \leq k$ at the node points. Suppose $\tilde{F}(x)$ contains complete polynomials of order k. Then the following interpolation error estimate holds[3,4]

$$\left\|\frac{d^m}{dx^m}(f-\tilde{F})\right\|_{L_q(0,L)} \equiv \left\{\int_0^L \left[\frac{d^m}{dx^m}(f-\tilde{F})\right]^q dx\right\}^{1/q} \leqq Kh^{k+1-m-\frac{1}{2}+(1/q)}\left\|\frac{d^{k+1}}{dx^{k+1}}f\right\|_{L_q(0,L)} \qquad (7)$$

where K is independent of h and $2 \leqq q \leqq \infty$. This inequality shall prove to be important in establishing convergence of the method or lack of it.

The Constrained Problem

It is not difficult to show that the procedure described above amounts to a finite element approximation of an entirely different functional than that stated in Eq. (1) for the original problem. Indeed, by replacing v' by η' in Eq. (4) and requiring that $\eta(x_i) = v(x_i)$, $i = 1, 2$, we have made it necessary to consider the new energy functional

$$\pi(u, v, \eta, \lambda) = \frac{1}{2}\int_0^L \left\{EIv''^2 + EAu'^2 + EAu'\eta'^2 + \frac{EA}{4}\eta'^4 - 2qv + 2\lambda(v-\eta)\right\} dx + Pu\Big|_0^L \qquad (8)$$

Here λ is a Lagrange multiplier associated with the constraint condition, $\eta - v = 0$.

For the approximate solution, we set $v = \boldsymbol{\phi}\mathbf{v}$, $u = \boldsymbol{\psi}\mathbf{u}$, $\eta = \boldsymbol{\psi}\boldsymbol{\eta} = \bar{\boldsymbol{\psi}}\mathbf{v}$, and $\lambda = \gamma(x)\boldsymbol{\lambda}$. Then, $\pi(\mathbf{u}, \mathbf{v}, \boldsymbol{\eta}, \boldsymbol{\lambda})$ assumes a stationary value for each finite element when the following local equations are satisfied:

$$\int_0^h \{EI\boldsymbol{\phi}''^T\boldsymbol{\phi}''\mathbf{v} - q\boldsymbol{\phi}^T + \boldsymbol{\phi}^T\gamma\boldsymbol{\lambda}\}\, dx = 0 \tag{9a}$$

$$\int_0^h \left\{EA\boldsymbol{\psi}'^T\boldsymbol{\psi}'\boldsymbol{\eta}\boldsymbol{\psi}'\mathbf{u} + \frac{EA}{2}\boldsymbol{\psi}'^T\boldsymbol{\psi}'\boldsymbol{\eta}\boldsymbol{\eta}^T\boldsymbol{\psi}'^T\boldsymbol{\psi}'\boldsymbol{\eta} - \boldsymbol{\psi}^T\gamma\boldsymbol{\lambda}\right\} dx = 0 \tag{9b}$$

$$\int_0^h \left\{EA\boldsymbol{\psi}'^T\boldsymbol{\psi}'\mathbf{u} + \frac{EA}{2}\boldsymbol{\psi}'^T\boldsymbol{\eta}^T\boldsymbol{\psi}'^T\boldsymbol{\psi}'\boldsymbol{\eta}\right\} dx = 0 \tag{9c}$$

$$\int_0^h \gamma^T(\boldsymbol{\phi}\mathbf{v} - \boldsymbol{\psi}\boldsymbol{\eta})\, dx = 0 \tag{9d}$$

Comparing Eqs. (9) and (6), we see that the latter can be obtained from Eq. (9) only by approximately satisfying the local constraint condition (9d). Since $\boldsymbol{\psi}(x)$ and $\boldsymbol{\phi}(x)$ are known in terms of h via Eq. (3), we can calculate immediately

$$\int_0^h \gamma^T(\boldsymbol{\phi}\mathbf{v} - \boldsymbol{\psi}\boldsymbol{\eta})\, dx = h/2\begin{Bmatrix}(v_1-\eta_1)+(v_2-\eta_2)\\(v_1-\eta_1)+(v_2-\eta_2)\end{Bmatrix} + \begin{Bmatrix}O(h^2)\\O(h^2)\end{Bmatrix} \tag{10}$$

assuming $\gamma(x) = [1, 1]$. Hence, introduction of Eq. (10) into Eq. (9) yields Eq. (6) plus terms of order h^2. However, this process only serves to interpret Eq. (6); it still remains to be shown that Eq. (6) is an acceptable approximation.

Convergence

Let u^*, v^*, η^*, and λ^* be the exact functions providing a stationary value of $\pi(u, v, \eta, \lambda)$, let U^*, V^*, and H^* be the finite element solution, and let $\tilde{U}$, $\tilde{V}$, and $\tilde{H}$ be finite element interpolations. Let u, v, η, and λ be arbitrary values of the indicated variables. Then a simple calculation leads to the relation

$$\begin{aligned}\pi(u,v,\eta,\lambda) = \pi(u^*,v^*,\eta^*,\lambda^*) + \int_0^L \Big\{&\frac{EA}{2}(u'-u^{*\prime})^2 + \\ &(EI/2)(v''-v^{*\prime\prime})^2 - (P/2)(\eta'-\eta^{*\prime})^2 + (EA/2)\eta^{*\prime 2}(\eta'-\eta^{*\prime})^2 + \\ &EA\eta^{*\prime}(\eta'-\eta^{*\prime})(u'-u^{*\prime}) + (EA/2)(u'-u^{*\prime})(\eta'-\eta^{*\prime})^2 + \\ &(EA/2)\eta^{*\prime}(\eta'-\eta^{*\prime})^3 + (EA/8)(\eta'-\eta^{*\prime})^4 + (\lambda-\lambda^*)(v-v^*) - \\ &(\lambda-\lambda^*)(\eta-n^*)\Big\}\, dx\end{aligned} \tag{11}$$

Among possible displacement fields the deformed beam can assume are those for which the terms appearing in the preceding integral are positive. This being the case, we can assert that

$$|\varepsilon| \leqq |\pi(\tilde{U}, \tilde{V}, \tilde{H}, \lambda^*) - \pi(u^*, v^*, \eta^*, \lambda^*)| \tag{12}$$

where ε is the error in energy

$$\varepsilon = \pi(U^*, V^*, H^*, \lambda^*) - \pi(u^*, v^*, \eta^*, \lambda^*) \tag{13}$$

Assuming that the slope is bounded (i.e. $|\eta^{*\prime}| < M_0 < \infty$), we can use the mean value theorem, and the Schwarz's inequality, to extract from Eqs. (11) and (12) the following inequality:

$$\begin{aligned}|\varepsilon| \leqq\; &(EA/2)\|\tilde{U}'-u^{*\prime}\|^2_{L_2(0,L)} + (EI/2)\|\tilde{V}''-v^{*\prime\prime}\|^2_{L_2(0,L)} - \\ &(P/2)\|\tilde{H}'-\eta^{*\prime}\|^2_{L_2(0,L)} + (EA/2)M_0{}^2\|\tilde{H}'-\eta^{*\prime}\|^2_{L_2(0,L)} + \\ &EAM_0\|\tilde{H}'-\eta^{*\prime}\|_{L_2(0,L)}\|\tilde{U}'-u^{*\prime}\|_{L_2(0,L)} + \\ &(EA/2)\|\tilde{U}'-u^{*\prime}\|_{L_\infty(0,L)}\|\tilde{H}'-\eta^{*\prime}\|^2_{L_2(0,L)} + \\ &(EA/2)M_0\|\tilde{H}'-\eta^{*\prime}\|_{L_\infty(0,L)}\|\tilde{H}'-\eta^{*\prime}\|^2_{L_2(0,L)} + \\ &(EA/8)\|\tilde{H}'-\eta^{*\prime}\|^2_{L_\infty(0,L)}\|\tilde{H}'-\eta^{*\prime}\|^2_{L_2(0,L)}\end{aligned} \tag{14}$$

Introducing Eq. (7) with $k = 1$ for approximations of u and η, $k = 3$ for those of v we obtain

$$\begin{aligned}|\varepsilon| \leqq\; &(EA/2)K_1h^2 + (EI/2)K_2h^4 - (P/2)K_3h^2 + (EA/2)M_0{}^2K_4h^2 + \\ &EAM_0K_5h^2 + (EA/2)K_6h^{5/2} + (EA/2)M_0K_7h^{5/2} + (EA/8)K_8h^3\end{aligned} \tag{15}$$

where in Eq. (15), $K_1, \ldots, K_8$ are constants independent of h. Clearly as $h \to 0$, $|\varepsilon| \to 0$.

Conclusions

When nonlinear effects are large, the constants K_i, $i \geqq 3$, appearing in Eq. (15) may be on the order of those obtained in error estimates for the linearized problem. Clearly, the use of cruder approximations for $\eta'(x)$ governs the rate-of-convergence of the model. Had cubic approximations been used throughout, we would have obtained $|\varepsilon| = O(h^4)$ for h sufficiently small; for the mixed problem, $|\varepsilon| = O(h^2)$. However, this still may be an acceptable rate of convergence, particularly if "mildly nonlinear" behavior is evident.

We remark that the use of simpler functions for such terms may well be justified in more complicated problems like plate and shell bending problems. In a typical nonlinear analysis where usually an iterative or a step by step method may be required to solve Eqs. (6a) and (6b), the terms $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ due to nonlinear effects need be calculated several times, so their calculations take major part of the solution effort. Simpler functions may avoid numerical integration of these terms and this may result with significant savings in computational efforts. However, a larger number of elements is required in an inconsistent model to obtain comparable accuracy.

References

[1] Oden, J. T., "Calculation of Geometric Stiffness Matrices for Stability Analysis of Complex Structures," *AIAA Journal*, Vol. 4, No. 8, Aug. 1966, pp. 1480–1482.

[2] Argyris, J. H., Kelsey, S., and Kamel, H., "Matrix Methods of Structural Analysis: A Précis of Recent Developments," *Matrix Methods of Structural Analysis*, AGARDograph 72, Pergamon Press, Oxford, England, 1964.

[3] Oden, J. T., *Finite Elements of Nonlinear Continua*, McGraw-Hill, New York, 1972.

[4] Schultz, M. and Varga, R. S., "L-Splines," *Numeriche Mathematik*, Vol. 10, No. 19, 1967.

A Graphical Method for the Investigation of Shock Interference Phenomena

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A NUMBER of recent studies of shock interference have been directed toward correlation and interpretation of local flow phenomena with special emphasis on the prediction of local heating rates.[1] Although the pressure-deflection-shock polar method of determining simple shock interference flowfields has been in hand many years,[2,3] the method is unwieldy and has never found wide acceptance as a design tool. Edney demonstrated its application to several relatively complex interference patterns[4] but his more recent work as well as that of others relies on machine programs which require access to sophisticated computers and are used successfully only after extensive experience by the operator.[5,6] Although the shock polar method lacks the arithmetic precision of the computer,

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